Definition: Antiderivative

Let $f(x)$ be a function, then an <u>Antiderivative</u> is a new function $F(x)$ such that $F'(x) = f(x)$.

That is to say, the antiderivative of a function is a new function whose derivative gives us our original $f(x)$ back.

<u>Example 1:</u>

Let $f(x) = e^{2x} + 3x^2 - 6x + 1$. Would $F(x) = \frac{e^{2x}}{2} + x^3 - 3x^2 + x$ be an antiderivative of $f(x)$?

Are there other antiderivatives of $f(x)$?

<u>Solution:</u>

We can check that $F'(x) = e^{2x} + 3x^2 - 6x + 1$. This means that $F(x)$ is an antiderivative of $f(x)$.

We note that there are many antiderivatives of $f(x)$ as when we derive a constant it always equals zero. This means we have the <u>general antiderivative</u> as: $F(x) = \frac{e^{2x}}{2} + x^3 - 3x^2 + x + \mathrm{c}$ where c is some unknown constant.

Strategy: Finding Antiderivatives Using Try and Derive Tables

How To Use it:	When To Use it:	Why this works?
When first finding an antiderivative of $f(x)$, we can try the following: 1) Think of a new function $F(x)$ that you will feel will have a derivative that will result in the given function $f(x)$. Place the $F(x)$ on the "try side" and determine the derivative and place it on the "derive side". 2) Derive the function $F(x)$ to see if you get $f(x)$. If it is only of by a coefficient, you can fix your coefficient by multiplying the coefficient on both sides. 3) Once you have the correct derivative equal to $f(x)$ then you have found your $F(x)$. 4) Do not forget to add the $+c$ at the end to account for the unknown constant. NOTE: We cannot multiply a variable on both sides to "fix" the antiderivative. Why? Deriving would result in a product rule (which odds are was not what you were considering)	When calculating an antiderivative of a function. Note that this strategy will only work on simple functions (small functions with small chain rules).	This is simply applying the definition of what it means to be an antiderivative. We must remember the general constant c as any constant always derives to become zero. We also must remember that coefficients can be ignored when deriving, so they can also be ignored when taking an antiderivative.

Examples: Antiderivatives

Example 2:

Determine the general antiderivative of $f(x) = e^{3x}$

Solution:

We set up our try and derive table. We know that deriving $e^{\square} = e^{\square}\square'$, so a good place to start to arrive at an e^{3x} would be e^{3x}:

Try	Derive
e^{3x}	$3e^{3x}$

We note that the derivative is off by a factor of $\frac{1}{3}$, but multiplying this would leave us with a coefficient (which does not impact the derivative) so multiplying both by $\frac{1}{3}$ gives us our desired result:

Fix the try	Derive
$\frac{1}{3}e^{3x}$	$\frac{1}{3}(3e^{3x}) = f(x)$

Thus our general antiderivative is $F(x) = \frac{1}{3}e^{3x} + c$

Examples: Antiderivatives

Example 3:

Determine the antiderivative of $f(x) = x^3 - x^2 + x - 1 + \frac{1}{x} - \sqrt{x}$

Solution:

We set up our try and derive table for each piece. We know that the derivative of $x^n = nx^{n-1}$, so going backwards, it is a good idea to try one exponent higher to get to the correct function, then see if we can fix the coefficient. (We also note that $\sqrt{x} = x^{1/2}$)

Try	Derive		Fix the try	Derive
x^4	$4x^3$	→	$\frac{x^4}{4}$	x^3
$-x^3$	$-3x^2$	→	$-\frac{x^3}{3}$	$-x^2$
x^2	$2x$	→	$\frac{x^2}{2}$	x
$-x$	-1			

For $1/x$, if we try to add one to the exponent of x^{-1} it gives us $x^0 = 1$, but we know it cannot be correct as the derivative of 1 is 0 and we can't fix it to become $1/x$. We should recall that $\ln(\mathrm{x})$ derives to become $1/x$. **IMPORTANT NOTE:** We actually showed before that $\ln(|x|)$ also derives to become $1/x$ as well. Since $\ln(|x|)$ allows for more values in the domain, we would prefer to have this as our antiderivative. This means:

Try	Derive		Fix the try	Derive
$\ln(|x|)$	$\frac{1}{x}$			
$-x^{3/2}$	$-\frac{3}{2}x^{1/2}$	→	$-\frac{2}{3}x^{3/2}$	$-x^{1/2}$

We can now put all of the pieces together to construct our antiderivative: $F(x) = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x + \ln(|x|) - \frac{2}{3}x^{3/2} + c$

List of Antiderivatives We Would Want to Remember:

If you prefer to memorize a list of antiderivatives, then the following will be helpful. Note that it would be more helpful to think of "what would I derive" as it will practice derivatives more and it will allow for less memorization for the final exam.

$f(x)$		$F(x)$	
x^n	$(n \neq 1)$	$\frac{x^{n+1}}{n+1} + c$	
$\frac{1}{x}$		$\ln\|x\| + c$	(why do we not need other log bases?)
$\frac{1}{ax+b}$		$\frac{\ln\|ax+b\|}{a} + c$	
$\sin(ax)$		$-\frac{\cos(ax)}{a} + c$	
$\cos(ax)$		$\frac{\sin(ax)}{a} + c$	
$\sec^2(ax)$		$\frac{\tan(ax)}{a} + c$	
e^{ax}		$\frac{e^{ax}}{a} + c$	
b^{ax}		$\frac{b^{ax}}{a\,ln(b)} + c$	
$\frac{1}{1+x^2}$		$\arctan(x) + c$	
$\frac{1}{\sqrt{1-x^2}}$		$\arcsin(x) + c$	(Why do we not need to remember arccos?)

Other functions/rules can be done, but typically require techniques that are more challenging (these techniques are covered in the follow up course like MATH 2007).

Definition: Initial Value Problem

Initial Value Problem is the process of solving for the unknown constant for the antiderivative.

How To Use it:	When To Use it:	Why this works?
When solving for the specific antiderivative we can: **1) Use antidifferentiation to find our general antiderivative (with the +c).** **2) Sub in the initial point into our antiderivative to find c.**	When you want to find the specific antiderivative (ie solve for "c") when given a function and a point on the original function.	This is simply using the understanding that a point that appears on a function means that the point can be subbed into the function and the left side and right side should remain equal.

Examples: Solving the Initial Value Problem

Example 4:

Solve the initial value problem when $f(x) = \frac{1}{1+25x^2} + \frac{1}{1-3x} + e^{2x}$ where $(0,5)$ appears on the original curve.

Solution:

We want our antiderivative first, so we set up our try and derive table.

Try	Derive		Fix the try	Derive
$\arctan(5x)$	$\frac{1}{1+25x^2}(5)$	→	$\frac{1}{5}(\arctan(5x))$	$\frac{1}{1+25x^2}$
$\ln(1-3x)$	$\frac{1}{1-3x}(-3)$	→	$-\frac{1}{3}(\ln\lvert 1-3x\rvert)$	$\frac{1}{1-3x}$
e^{2x}	$2e^{2x}$	→	$\frac{1}{2}e^{2x}$	e^{2x}

This means $F(x) = \arctan(5x) - \frac{1}{3}\ln(1-3x) + \frac{1}{2}e^{2x} + c$

To solve the initial value problem, we solve for c by subbing in $(0,5)$

$$5 = \frac{1}{5}\arctan(5(0)) - \frac{1}{3}\ln|1-3(0)| + \frac{1}{2}e^{2(0)} + c$$

$$5 = 0 - 0 + \frac{1}{2} + c$$

$$c = \frac{9}{2}$$

$$\therefore F(x) = \frac{1}{5}\arctan(5x) - \frac{1}{3}\ln|1-3x| + \frac{1}{2}e^{2x} + \frac{9}{2}$$

Examples: Solving the Initial Value Problem

Example 5:

Let $\frac{d^2y}{dx^2} = 18x + 6 - 8e^{2x}$. Solve the initial value problem if $y(0) = 1$ and $y'(0) = -6$.

Solution:

We want our original function and we want to solve for the unknown constants that appear. Since we are given a second derivative, we will need to do two antiderivatives. Using our formulas (or our try and derive tables) we should see:

$$\frac{dy}{dx} = 9x^2 + 6x - 4e^{2x} + c$$

If we antiderive once more we should get:

$$y = 3x^3 + 3x^2 - 2e^{2x} + cx + d$$

Now we sub in our information to get our unknown constants. Subbing in -6 into our derivative and 1 into our original function gives:

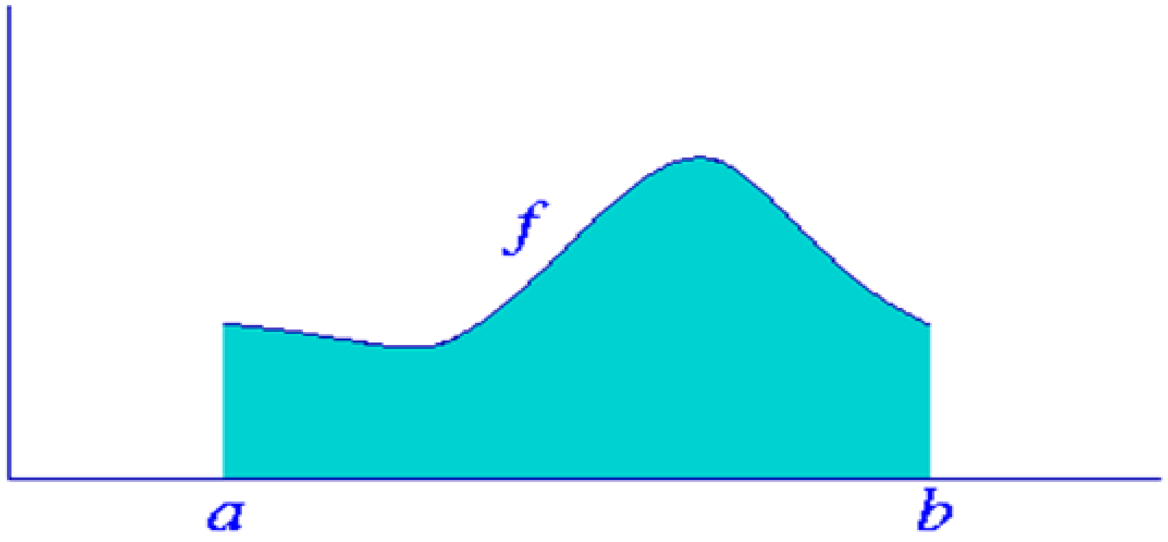
$$-6 = 9(0)^2 + 6(0) - 4e^{2(0)} + c \quad \rightarrow -6 = -4 + c \quad \rightarrow c = -2$$

$$1 = 3(0)^3 + 3(0)^2 - 2e^{2(0)} - 2(0) + d \quad \rightarrow 1 = -2 + d \quad \rightarrow d = 3$$

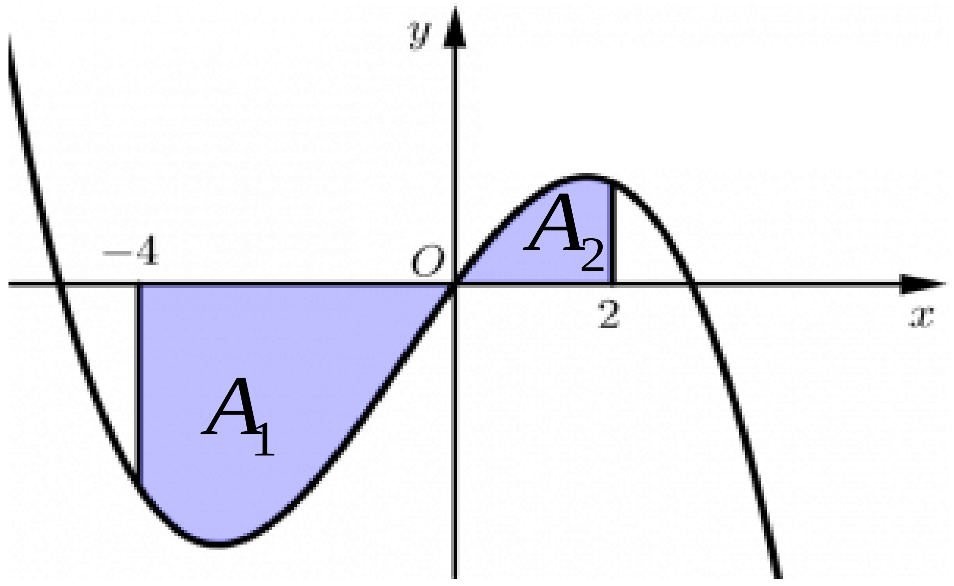
$$\therefore y = 3x^3 + 3x^2 - 2e^{2x} - 2x + 3$$

Definition: Integral

Consider a function $f(x)$. Assuming that $f(x)$ is not the x-axis, then there would be an area that the function makes with the x-axis. For example:



We define an "integral of f(x) from a to b with respect to x" as the amount of area above the x-axis but below the curve of f(x). We use the symbol: $\int_a^b f(x)dx$ to represent this notion.



We further note that if a function appears below the x-axis, then this area would be represented as a negative. That is A_1 would be a negative area, and A_2 would be a positive area. This also means that $\int_{-4}^{2} f(x)dx = A_2 - A_1$. This is to say that an integral calculates the net area (positive – negative area).

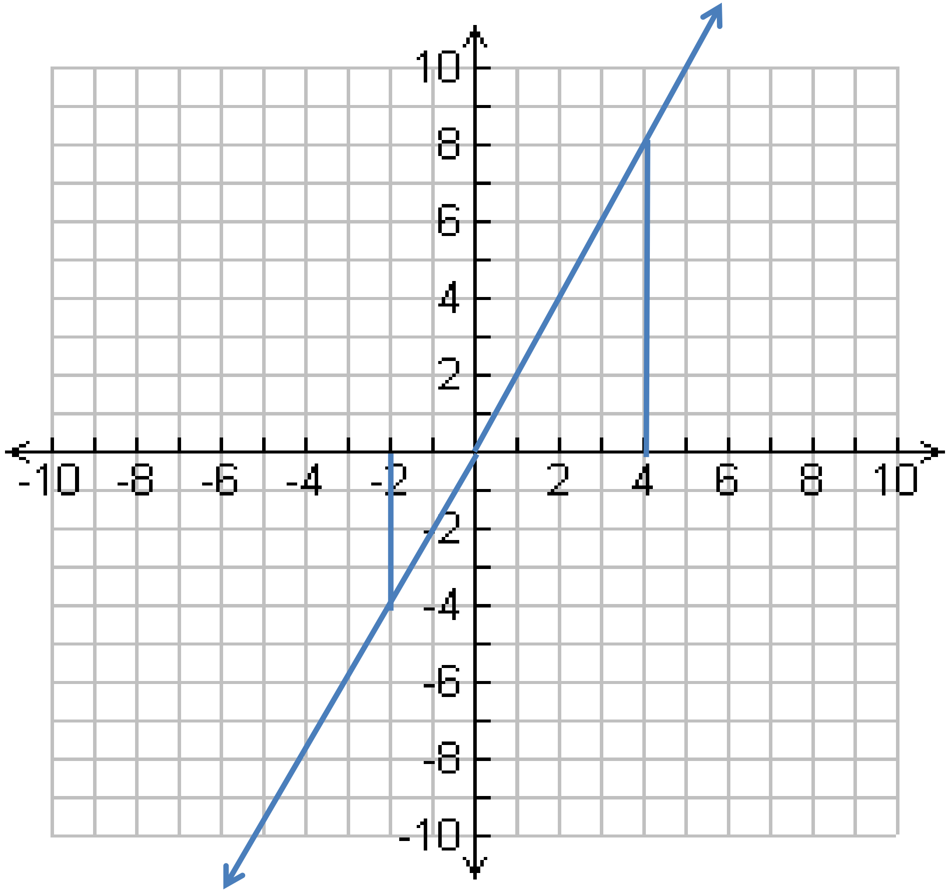
Examples: Integrals

Example 1:

Evaluate the integral a) $\int_0^4 2x\,dx$ and b) $\int_{-2}^4 2x\,dx$ and

Solution:

We would note that if we drew the function we would get:



a) Since an integral asks for the net area made with the x-axis, this would give us the area of the shaded triangle. Since the triangle is above the x-axis, this would be a positive area, thus we get:

$$\begin{aligned}\int_0^4 2x\,dx &= \frac{bh}{2}\\ &= \frac{4(8)}{2}\\ &= 16\end{aligned}$$

b) For this integral, we know that the small triangle from $[-2,0]$ is below the x-axis. So this is a negative area. It is still the area of a triangle (this time with area = 4). Thus we get a net area calculation:

$$\begin{aligned}\int_{-2}^4 2x\,dx &= A_+ - A_-\\ &= 16 - 4\\ &= 12\end{aligned}$$

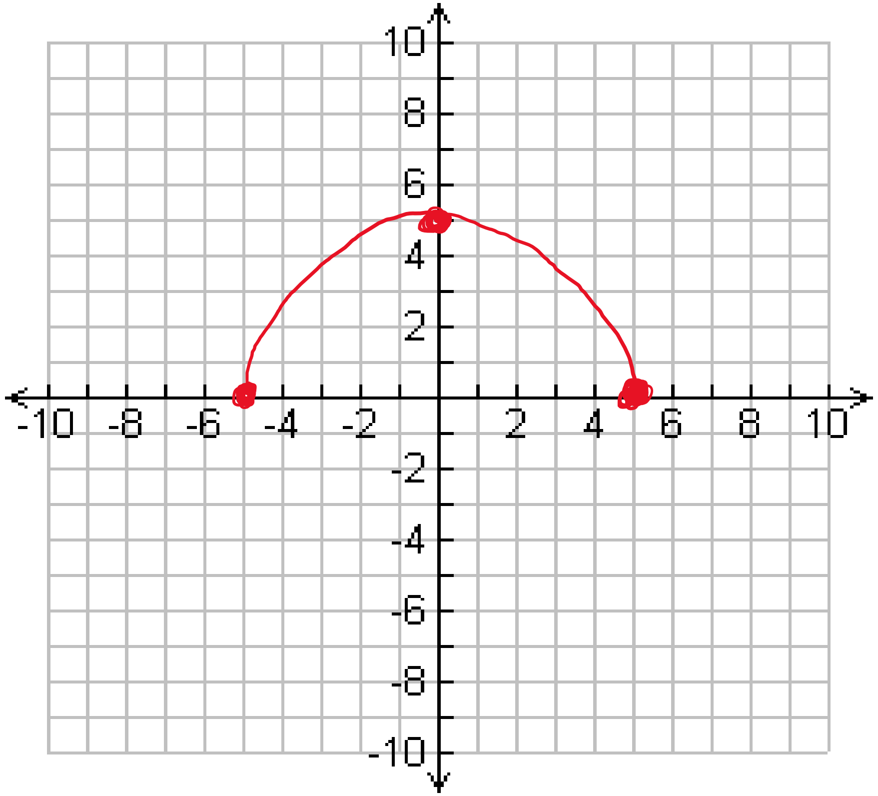
Examples: Integrals

Example 1:

Evaluate the integral $\int_0^5 \sqrt{25 - x^2}\, dx$

Solution:

If we recall the equation of a circle $x^2 + y^2 = r^2$ and rearrange for $y = \pm\sqrt{r^2 - x^2}$, we should see that $\sqrt{25 - x^2}$ is the equation for the top half of a circle with radius of 5:



Since the integral is asking for an area of this circle from 0 to 5, it is actually asking for the area of a quarter circle of radius 5:

$$\int_0^5 \sqrt{25 - x^2}\, dx = \frac{\pi r^2}{4} = \frac{\pi(5)^2}{2} = \frac{25\pi}{2}$$

Examples: Integrals

Example 2:

Evaluate the integral $\int_3^3 e^x dx$

Solution:

Although you may be worried about finding the area underneath an exponential curve, in this case, if we check the bounds, we should see that the width of the area we are asking to find is $3 - 3 = 0$. An area of 0 with will always have a 0 area, so we get $\int_3^3 e^x dx = 0$

Theorem: Fundamental Theorem of Calculus

Conditions:

Let f be continuous on $[a, b]$.

Let $F(x) = \int_a^x f(t)dt$

Results:

$F(x)$ **is a continuous function on** $[a, b]$

$F(x)$ **is differentiable on** (a, b)

$F'(x) = f(x)$

$\int_a^b f(t)dt \; = \; F(b) - F(a)$

Essentially this tells us that the integral (area under a curve) for a continuous function is actually the same as taking the antiderivative of the function.

How To Use it:

When calculating an integral from $\int_a^b f(t)dt$:

1) **Ensure the function is continuous n [a,b]**
2) **Find the antiderivative $F(x)$**
3) **Determine $F(b) - F(a)$ which is the value of our integral.**

When To Use it:

When calculating integrals (then we should think that this is an antiderivative). We should also note that we use this when we want to find areas under curves (made with the x-axis) which we will see more applications of this in our next few lessons.

Why this works?

You can see a proof here. This is a tougher proof that requires some understanding of other theorems.

Examples: Fundamental Theorem of Calculus

Example 3:

Calculate the integral: $\int_1^8 \frac{x+2x^2-1}{x}\, dx$

Solution:

We note that our function is continuous on $[1,8]$ so we can use the fundamental theorem of calculus to find the integral by calculating an antiderivative. Before we find the antiderivative, we note that we have not covered any rules that allows us to antiderive a division. So we should manipulate the expression to calculate the antiderivative:

$$\int_1^8 \frac{x+2x^2-1}{x}\, dx = \int_1^8 \frac{x}{x} + \frac{2x^2}{x} - \frac{1}{x}\, dx$$

$$= \int_1^8 1 + 2x - \frac{1}{x}\, dx$$

We can then use our "try and derive method" to antidifferentiate (or recall our rules). Once we antiderive, we will get:

$$F(x) = x + x^2 - \ln|x| + c$$

Finally, we can find $F(b) - F(a)$ to solve our integral:

$$F(b) = F(8) = 72 - \ln 8 + c$$

$$F(a) = F(1) = 2 + c$$

$$\therefore \int_1^8 \frac{x+2x^2-1}{x}\, dx = F(8) - F(1) = 70 - \ln(8)$$

Examples: Fundamental Theorem of Calculus

Example 4:

Determine: $\frac{d}{dx}\left(\int_3^{x^2} \frac{\sin(t)}{\ln t\,(\arctan(t))}\,dt\right)$ where $x > 3$

Solution:

At first glance, this looks impossible as we have not even learn tools on how to handle antiderivatives of complicated functions with products and quotients, but this is actually not too bad of a question if we understand the fundamental theorem of calculus.

We know that $f(t) = \frac{\sin(t)}{\ln t\,(\arctan(t))}$ is a continuous function on the interval $[3, x^2]$ as we do not end up with divisions by 0, or ln of 0 or negatives as long since our inputs are higher than 3.

This means that we know there is some antiderivative $F(t)$, even though we do not know what it looks like. Thus we can apply :

$$\begin{aligned}
\frac{d}{dx}\left(\int_3^{x^2} \frac{\sin(t)}{\ln t\,(\arctan(t))}\,dt\right) &= \frac{d}{dx}\big(F(x^2) - F(3)\big) \\
&= F'(x^2)[2x] - F'(3)([0]) && \text{(from the chain rule)} \\
&= 2x\ \ f(x^2) && \text{(since F is the antiderivative of f)} \\
&= 2x\left(\frac{\sin(x^2)}{\ln x^2\,(\arctan(x^2))}\right) && \text{(our f function)}
\end{aligned}$$

Examples: Fundamental Theorem of Calculus

Example 5:

Explain why we cannot use the fundamental theorem of calculus directly on the following integral:

$$\int_{-1}^{1} \frac{1}{x}\, dx$$

Solution:

We cannot use the fundamental theorem of calculus on this integral as the function is not continuous on [-1,1]. We call this an improper integral.

Definition: Indefinite and Definite Integrals

A definite integral denoted by $\int_a^b f(x)dx = F(b) - F(a)$ is the notation we use when we want to find the net area from a to b made with the x axis. (Other applications exist as well (determining probability, length of curves, volumes, etc…), but the focus in this class will be areas).

The indefinite integral denoted by $\int f(x)dx = F(x) + c$ is the notation that we use when we simply want to find the general antiderivative.

We note that this means that when calculating a definite integral, you do not need to concern yourself with the $+c$ (as they will simply cancel out with the subtraction of $F(b) - F(a)$).

Examples: Indefinite and Definite Integrals

Example 6:

Determine a) $\int -6x^2 + 8x - e^{-2x}\,dx$ b) $\int_0^1 -6x^2 + 8x - e^{-2x}\,dx$

Solution:

a) Since this is an indefinite integral, we simply want to calculate the general antiderivative:

Try	Derive		Fix the try	Derive
x^3	$3x^2$	→	$-2x^3$	$-6x^2$
x^2	$2x$	→	$4x^2$	$8x$
e^{-2x}	$-2e^{-2x}$	→	$\frac{1}{2}e^{-2x}$	$-e^{-2x}$

Thus $\int -6x^2 + 8x - e^{-2x}\,dx = -2x^3 + 4x^2 + \frac{1}{2}e^{-2x} + c$

b) We note that we can use the FTC as our initial function is continuous on [0,1]. A definite integral gives us:

$$
\begin{aligned}
F(b) - F(a) &= F(1) - F(0) \\
&= -2(1)^3 + 4(1)^2 + \tfrac{1}{2}e^{-2(1)} - \left[-2(0)^3 + 4(0)^2 + \tfrac{1}{2}e^{-2(0)}\right] \\
&= -2 + 4 + \frac{1}{2e^2} - \left[\frac{1}{2}\right] \\
&= \frac{3}{2} + \frac{1}{2e^2}
\end{aligned}
$$

Examples: Indefinite and Definite Integrals

Example 7:

Determine a) $\int \tan^2(\theta)\ dx$ b) $\int_0^{\frac{\pi}{4}} \tan^2(\theta)\, dx$

Solution:

a) Since this is an indefinite integral, we simply want to calculate the general antiderivative. At first glance, this looks very difficult (as we have not even seen how to handle "products" or "chains" for integrals). However, we should think of identities as they can help us solve this problem:

$$\sin^2 x + \cos^2 x = 1 \quad \rightarrow \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\rightarrow \tan^2 x + 1 = \sec^2 x$$

$$\rightarrow \tan^2 x = \sec^2 x - 1$$

This allows us to find the antiderivative as we know $\tan x$ derives to become $\sec^2 x$:

$$\int \tan^2(\theta)\ dx = \int \sec^2 x - 1\, dx$$

$$= \tan(x) - x + c$$

b) We note that we can use the FTC as our initial function is continuous on $[0,\frac{\pi}{4}]$. A definite integral gives us:

$$F(b) - F(a) = F\left(\frac{\pi}{4}\right) - F(0)$$

$$= \tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4} - [\tan(0) - 0]$$

$$= 1 - \frac{\pi}{4}$$

Formula: Net Area vs Total Area

Recall that $\int_a^b f(x)$ calculates the **net area** starting from $x = a$ to $x = b$ made with the x −axis. That is:

$$\int_a^b f(x) = (All\ Area\ Above\ x - axis\ on [a, b]) - (All\ Area\ Below\ x - axis\ on\ [a, b])$$

If we wish to calculate the **total area** we can instead use the following strategy:

How To Use it:	When To Use it:	Why this works?
1) Find all x intercepts of the function inside the interval. (Let us say we have 3 of them, call them x_1, x_2, and x_3.) 2) Our total area would be: Total Area = $\left\|\int_a^{x_1} f(x)dx\right\| + \left\|\int_{x_1}^{x_2} f(x)dx\right\| + \left\|\int_{x_2}^{x_3} f(x)dx\right\| + \left\|\int_{x_3}^{b} f(x)dx\right\|$ Note that if we have more x-intercepts, then we would continue to break up the integral over more pieces which expands the above formula.	When calculating the total area (not the net area) made with the x-axis and the function f(x)	By splitting the interval up based on the x intercepts, we are guaranteeing that the pieces are either all positive area or all negative area. If we then take the absolute value of each piece, we then force all pieces to be positive and can add them together.

Examples: Net Area vs Total Area

Example 1:

Determine the a) net area and b) the total area made with the x-axis of the function below over the interval $[-2,10]$:

$f(x) = (x)(x-4)(x+4)$

Solution:

a) To find the net area, we simply calculate the integral $\int_{-2}^{10}(x)(x-4)(x+4)\ dx$.

$$\int_{-2}^{10}(x)(x-4)(x+4)\,dx \qquad = \int_{-2}^{10} x^3 - 16x\,dx$$

We note that the function is continuous on the interval, so we can calculate the area by finding the antiderivative (note we do not need the +c as we will be cancelling them out when subtracting): $F(x) = \frac{x^4}{4} - 8x^2$

$$\begin{aligned} \therefore \int_{-2}^{10} x^3 - 16x\,dx \qquad &= F(10) - F(-2) \\ &= \frac{10000}{4} - 8(100) - \left[\frac{16}{4} - 8(4)\right] \\ &= 2500 - 800 - [-28] \\ &= 1728 \end{aligned}$$

Examples: Net Area vs Total Area

Example 1 (Continued):

Determine the a) net area and b) the total area made with the x-axis of the function below over the interval $[-2,10]$:

$$f(x) = (x)(x-4)(x+4)$$

Solution:

b) To find the total area, first need to find the x intercepts then split our integral pieces and use absolute values to grantee positive areas. Since our expression is already factored, we should see that the zeros are -4, 0, and 4. Only 2 of these appear in the interval [-2,10] so our total area formula becomes:

$$\text{Total area} = \left|\int_{-2}^{0}(x)(x-4)(x+4)\ dx\right| + \left|\int_{0}^{4}(x)(x-4)(x+4)\ dx\right| + \left|\int_{4}^{10}(x)(x-4)(x+4)\ dx\right|$$

Our the antiderivative doesn't change (as we do not change the function), so we can still use: $F(x) = \frac{x^4}{4} - 8x^2$

Total area $= |F(0) - F(-2)| + |F(4) - F(0)| + |F(10) - F(4)|$

It is probably best to calculate these endpoints separately just to be careful.

$$F(-2) = \frac{(-2)^4}{4} - 8(-2)^2 = -28$$

$$F(0) = \frac{(0)^4}{4} - 8(0)^2 = 0$$

$$F(4) = \frac{(4)^4}{4} - 8(4)^2 = -64$$

$$F(10) = \frac{(10)^4}{4} - 8(10)^2 = 1700$$

$$\begin{aligned} \therefore \text{Total area} &= |F(0) - F(-2)| + |F(4) - F(0)| + |F(10) - F(4)| \\ &= |0 - (-28)| + |-64 - (0)| + |1700 - (-64)| \\ &= 28 + 64 + 1764 \\ &= 1856 \end{aligned}$$

Examples: Net Area vs Total Area

Example 2:

Determine the a) net area and b) the total area made with the x-axis of the function below over the interval $[1, e^2]$:

$f(x) = \frac{e}{x} - 1$

Solution:

a) To find the net area, we simply calculate the integral $\int_1^{e^2} \frac{e}{x} - 1 \; dx$:

We note that the function is continuous on the interval, so we can calculate the area by finding the antiderivative (note we do not need the +c as we will be cancelling them out when subtracting):

$$F(x) = e\ln|x| - x$$

$$\begin{aligned}
\therefore \int_1^{e^2} \frac{e}{x} - 1 \quad &= F(e^2) - F(1) \\
&= e\ln(e^2) - e^2 - [e\ln(1) - 1] \\
&= 2e - e^2 - [-1] \\
&= 1 + 2e - e^2
\end{aligned}$$

Examples: Net Area vs Total Area

Example 1 (Continued):

Determine the a) net area and b) the total area made with the x-axis of the function below over the interval $[1, e^2]$:

$f(x) = \frac{e}{x} - 1$

Solution:

b) To find the total area, first need to find the x intercepts then split our integral pieces and use absolute values to grantee positive areas.

We solve for the zeros first: $\frac{e}{x} - 1 = 0 \quad \rightarrow \quad \frac{e}{x} = 1 \quad \rightarrow \quad \mathrm{e} = x$

Since e is in the interval, we will need to split up our integral over this zero to find the total area:

$$\text{Total area} = \left|\int_1^e \frac{e}{x} - 1 \; dx\right| + \left|\int_e^{e^2} \frac{e}{x} - 1 \; dx\right|$$

Our the antiderivative doesn't change (as we do not change the function), so we can still use: $F(x) = e\ln|x| - x$

Total area $= |F(e) - F(1)| + |F(e^2) - F(e)|$

It is probably best to calculate these endpoints separately just to be careful.

$F(1) = e\ln(1) - 1 = -1$

$F(e) = e\ln(e) - e = 0$

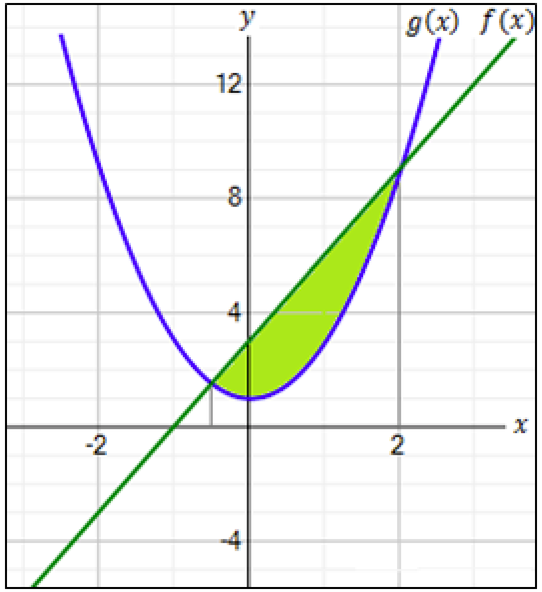
$F(e^2) = e\ln(e^2) - e^2 = 2e - e^2$

$$\begin{aligned}\therefore\text{Total area} &= |F(e) - F(1)| + |F(e^2) - F(e)| \\ &= |0 - (-1)| + |2e - e^2 - 0| \\ &= |1| + |2e - e^2|\end{aligned}$$

We know that $e \sim 2.7$, so we know that $2e < e^2$. This means that $2e - e^2 < 0$ so $|2e - e^2| = -(2e - e^2)$ or simply $e^2 - 2e$. Thus our total area is: $e^2 - 2e + 1$

Formula: Area Contained Between Two Curves

An area contained between two curves means the area that is enclosed by two (or more) points of intersection between the two functions:



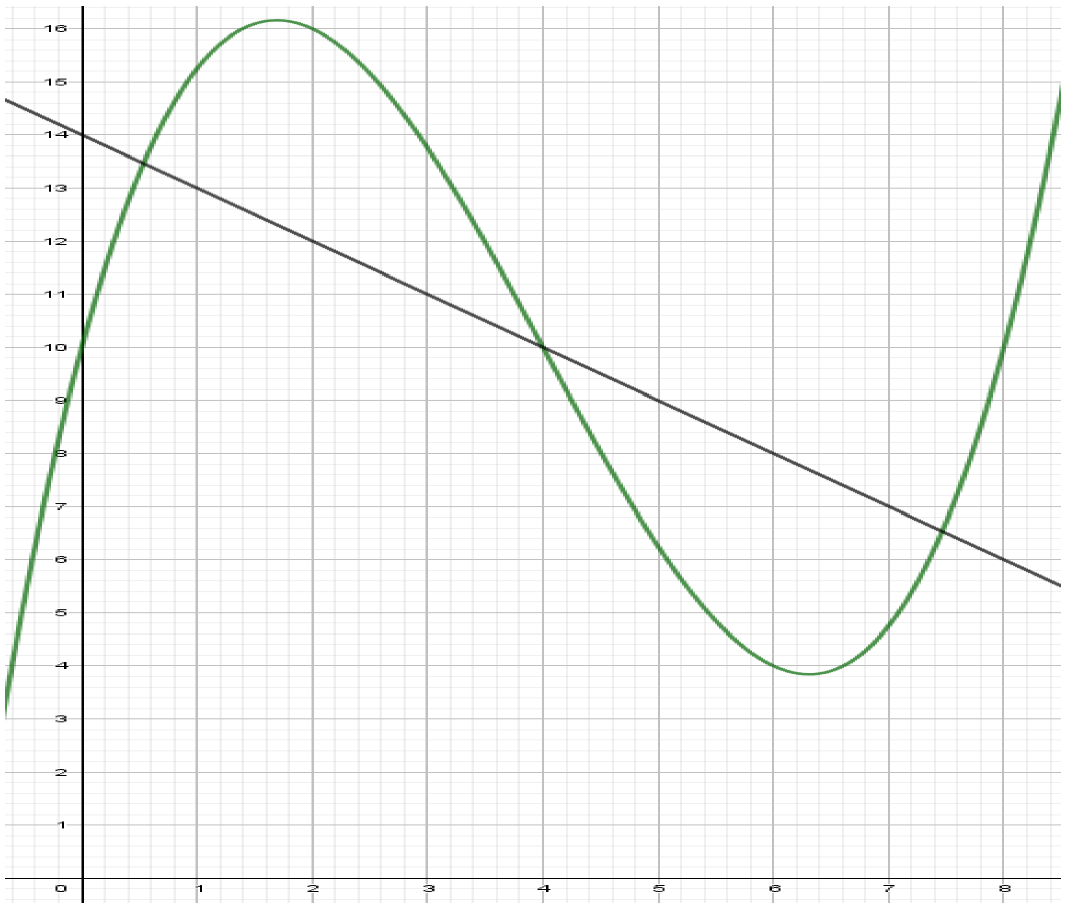
How To Use it:	When To Use it:	Why this works?
1) Find all x-values for the points of intersection between the two curves (Let us say we have 4 of them, call them x_1, x_2, x_3, and x_4.) 2) Our contained area would be: Contained Area = $\left\lvert\int_{x_1}^{x_2} f(x) - g(x)\ dx\right\rvert + \left\lvert\int_{x_2}^{x_3} f(x) - g(x)\ dx\right\rvert + \left\lvert\int_{x_3}^{x_4} f(x) - g(x)\ dx\right\rvert$ Note that if we have more x-intercepts, then we would continue to break up the integral over more pieces which expands the above formula.	When calculating the total area made between two functions f(x) and g(x).	See proof section.

Examples: Area Contained Between Two Curves

Example 3:

a) Shade the area for: $\int_{0.5}^{7.5} f(x)\ dx$ and $\int_{0.5}^{7.5} g(x)\ dx$ in the diagram below (where f(x) is the cubic function, and g(x) is the line)

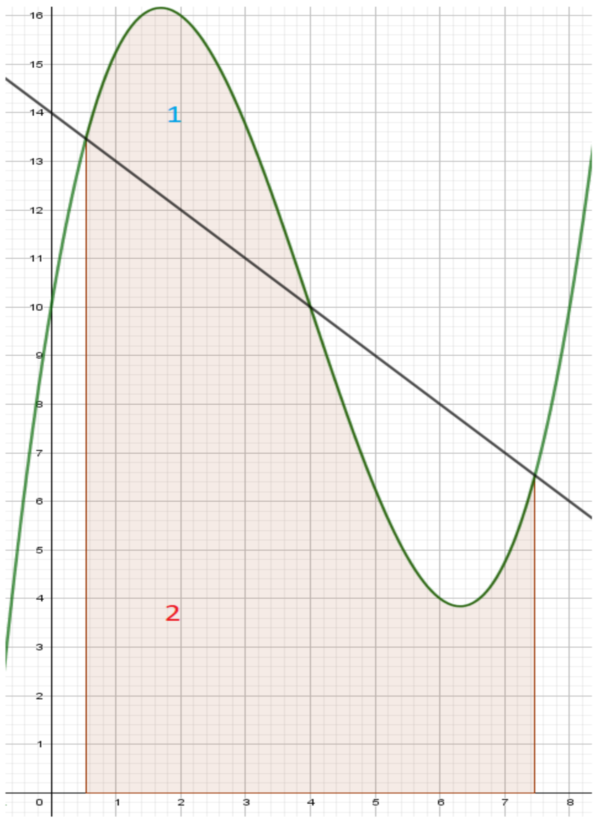
b) Describe the area for $\int_{0.5}^{7.5} f(x) - g(x)\ dx$ and explain what you notice and why we must consider all points of intersection when we are finding the area contained between two curves.

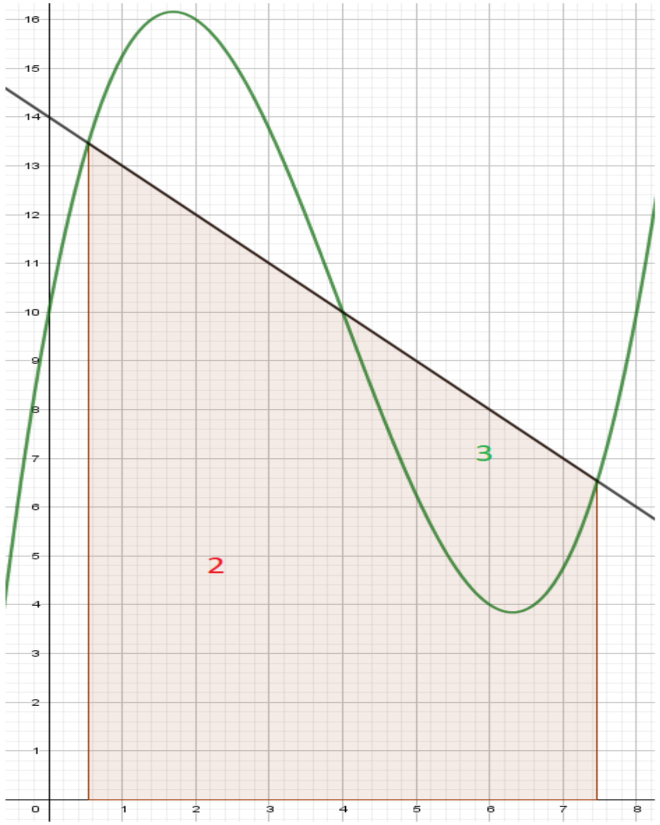


Examples: Area Contained Between Two Curves

Example 3 (continued):

a) $\int_{0.5}^{7.5} f(x)\ dx$ $\int_{0.5}^{7.5} g(x)$





b) If we then subtract these two from each other, we should see that the "Area 2" will cancel out, and we will have

$$\int_{0.5}^{7.5} f(x) - g(x)\ dx = A_1 - A_3$$

c) We were hoping to find $A_1 + A_3$, so if we ignore the centre intersection point it does not give us what we want. In fact, the only way it would give us what we want is if f was always higher than g. Since we cannot guarantee this (and we want to avoid graphing the functions every time if possible) we simply consider all of the points of intersection separately to ensure that we calculate the correct area, then take the absolute value of each piece to ensure that all areas remain positive.

Examples: Area Contained Between Two Curves

Example 4:

Determine the area contained between the curves $f(x) = x^3$ and $g(x) = 2x^2$

Solution:

We first find x-values for the points of intersection:

$f(x) = g(x)$

$x^3 = 2x^2$

$x^3 - 2x^2 = 0$

$x^2(x - 2) = 0$

Thus either $x = 0$ or $x = 2$

Since we only have two points of intersection, our contained area can be found as:

$$\left|\int_{x_1}^{x_2} f(x) - g(x)\, dx\right| = \left|\int_0^2 x^3 - 2x^2\, dx\right|$$

Our function is continuous, so we can use the FTC. We know the antiderivative $A(x) = \frac{x^4}{4} - \frac{2x^3}{3}$ so we get

$$|A(2) - A(0)| = \left|\frac{2^4}{4} - \frac{2(2)^3}{3} - \left(\frac{0^4}{4} - \frac{2(0)^3}{3}\right)\right|$$

$$= \left|4 - \frac{16}{3}\right|$$

$$= \left|-\frac{4}{3}\right|$$

$$= \frac{4}{3}$$

Examples: Area Contained Between Two Curves

Example 5:

Determine the area contained between the curves $f(\theta) = \sin(\theta)$ and $g(\theta) = \cos(\theta)$ between the first two points of intersection only found within the domain of $[0,2\pi]$

Solution:

We first find θ-values for the points of intersection:

$$f(\theta) = g(\theta)$$

$$\sin(\theta) = \cos(\theta)$$

$$\frac{\sin(\theta)}{\cos(\theta)} = 1$$

$$\tan(\theta) = 1$$

$\therefore$ Within our unit circle, we note that $\tan(\theta) = 1 = \frac{y}{x}$ only at two points within the first full circle at $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$

Since we only have two points of intersection, our contained area can be found as: $\left|\int_{x_1}^{x_2} f(x) - g(x)\, dx\right| = \left|\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin(\theta) - \cos(\theta) \;\; d\theta\right|$

Our function is continuous, so we can use the FTC. We know the antiderivative $A(\theta) = -\cos(\theta) - \sin(\theta)$ so we get

$$\begin{aligned}
\left|A\left(\frac{5\pi}{4}\right) - A\left(\frac{\pi}{4}\right)\right| &= \left|-\cos\left(\frac{5\pi}{4}\right) - \sin\left(\frac{5\pi}{4}\right) - \left(-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right)\right| \\
&= \left|-\left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right) - \left(-\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\right)\right| \\
&= \left|\frac{4\sqrt{2}}{2}\right| \\
&= 2\sqrt{2}
\end{aligned}$$

Examples: Area Contained Between Two Curves

Example 6:

Determine the area contained between the curves $f(x) = x^3 + x^2$ and $g(x) = x^2 + 4x$

Solution:

We first find x-values for the points of intersection:

$f(x) = g(x)$

$x^3 + x^2 = x^2 + 4x$

$x^3 - 4x = 0$

$x(x^2 - 4) = 0$

Thus either $x = -2, 0$ or 2.

Since we have three points of intersection, our contained area can be found as:

$$\left|\int_{x_1}^{x_2} f(x) - g(x)\, dx\right| + \left|\int_{x_2}^{x_3} f(x) - g(x)\, dx\right| \qquad = \left|\int_{-2}^{0} x^3 - 4x\, dx\right| + \left|\int_{0}^{2} x^3 - 4x\, dx\right|$$

Our function is continuous, so we can use the FTC. We know the antiderivative $A(x) = \frac{x^4}{4} - 2x^2$ so we get

$$= |A(0) - A(-2)| + |A(2) - A(0)|$$

$$= \left|\frac{0^4}{4} - 2(0)^2 - \left(\frac{(-2)^4}{4} - 2(-2)^2\right)\right| + \left|\frac{(2)^4}{4} - 2(2)^2 - \left(\frac{0^4}{4} - 2(0)^2\right)\right|$$

$$= |4| + |-4|$$

$$= 8$$